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A remark on Serre's example of p-adic Eisenstein series

by

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1 Introduction.

In [Se], J. P. Serre developed the theory of p-adic modular forms and applied it to the construction of p-adic zeta function. In this paper, we shall try to generalize a formula for p-adic Eisenstein series which was originally given by Serre. A p-adic modular form is a formal power series

$$f=\sum_{t=0}^{\infty}a(t)\,q^{t}\in\mathbb{Q}_{p}[[q]]$$

which is the limit of a sequence of modular forms $\{f_m\}$ with rational Fourier coefficients: $\lim_{m\to\infty}f_m=f$.
If we denote by

$$f_m=\sum_{t=0}^{\infty}a^{(m)}(t)\,q^{t}\in\mathbb{Q}[[q]]$$

the Fourier expansion of f_m (q-expansion), this limit means that

$$v_p(f-f_m):=\inf_t v_p(a(t)-a^{(m)}(t))\to+\infty\quad(m\to\infty),$$

where v_p is the valuation of $\mathbb{Q}_p$ normalized as $v_p(p)=1$. If we denote by $\{k_m\}$ the weight of $\{f_m\}$, then Serre showed that $\{k_m\}$ has the limit in the following set:

$$X:=\varprojlim X/(p-1)p^{m-1}\mathbb{Z}=\mathbb{Z}_p\times\mathbb{Z}/(p-1)\mathbb{Z}\,.$$

Let $E_k^{(n)}$ be the Siegel-Eisenstein series of degree n and weight k (for precise definition, see §2). Set

$$G_k:=\frac{1}{2}\,\zeta(1-k)\,E_k^{(1)},$$

where $\zeta(s)$ is the Riemann zeta function. For $k\in X$, we take a sequence $\{k_m\}\subset 2\mathbb{Z}$ such that $\lim_{m\to\infty}k_m=k$ and $|k_m|\to+\infty\,(m\to\infty)$. Serre defined the p-adic Eisenstein series G_k^{*} of weight $k\in X$ by

$$G_k^{*}:=\lim_{m\to\infty}G_{k_m}.$$

The right-hand side converges and it becomes a p-adic modular form. The following example is due to Serre:

EXAMPLE of G_k^*. *let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$ and $k = \left(1, \frac{p+1}{2}\right) \in X$. Then we have*

$$G_k^* = h(-p) + \sum_{t=1}^{\infty} \sum_{0<d|t} \left(\frac{d}{p}\right) q^t,$$

where $h(-p)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

The main purpose of this paper is to give a generalization of this example. The Siegel modular form $f(Z)$ has a Fourier expansion of the form

$$f(Z) = \sum_T a_f(T) \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)] = \sum_T a_f(T)\, q^T,$$

where T runs over the set of half-integral, positive semi-definite symmetric matrices (see §2). For $T = (t_{ij})$ and $Z = (z_{ij})$, we set $q_{ij} := \exp(2\pi\sqrt{-1}\, z_{ij})$, $q_i = q_{ii}$, and $t_i = t_{ii}$. Then f can be regarded as a power series in $\mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \cdots, q_n]]$. So we can define the p-adic Siegel modular form as an element of $\mathbb{Q}[q_{ij}, q_{ij}^{-1}]$ $[[q_1, \cdots, q_n]]$. Our result can be stated as follows:

THEOREM *Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$. If we put*

$$k_m := 1 + \frac{p-1}{2} \cdot p^{m-1} \in \mathbb{Z},$$

then the sequence $\{k_m\}$ has the limit $k = \left(1, \frac{p+1}{2}\right) \in X$ and

$$\begin{aligned} E_k^* &:= \lim_{m\to\infty} \left(\frac{1}{2}\zeta(1-k_m)\, E_{k_m}^{(2)}\right) \\ &= \frac{1}{2} h(-p) + \sum_{\substack{T\geq 0 \\ D(T)=-p\ or\ 0}} \mathrm{rank}\,(T) \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) q^T, \end{aligned}$$

where $D(T)$ is the discriminant of the field $\mathbb{Q}\left(\sqrt{-\det(2T)}\right)$ and we understand $D(T) = 0$ if $\det(T) = 0$, and $\varepsilon(T) := \mathrm{g.c.d}\,(t_{11}, 2t_{12}, t_{22})$.

In the final section, we give an additional formula which is concerned with reduction mod p of the Fourier coefficient of the Siegel-Eisenstein series.

2 Siegel-Eisenstein series.

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n:

$$\mathbb{H}_n := \{Z = X + \sqrt{-1}Y \in \mathrm{Sym}_n(\mathbb{C}) \mid Y > 0\}.$$

The real symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acts on $\mathbb{H}_n$ by

$$Z \longmapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}).$$

The group $\Gamma_n := \mathrm{Sp}_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ is called the Siegel modular group. Let $[\Gamma_n, k]$ denote the $\mathbb{C}$-vector space of Siegel modular forms of weight k for Γ_n. Any element f in $[\Gamma_n, k]$ admits a Fourier expansion of the form

$$f(Z) = \sum_{0 \le T \in \Lambda_n} a_f(T) \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)], \tag{2.1}$$

where the index set Λ_n is defined by

$$\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii} \in \mathbb{Z},\ 2t_{ij} \in \mathbb{Z}\}. \tag{2.2}$$

Let $\Gamma_{n,0}$ be the subgroup of Γ_n defined by

$$\Gamma_{n,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C = O_n \right\}.$$

For an even integer k, we define a series

$$E_k^{(n)}(Z) := \sum_{\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_{n,0} \backslash \Gamma_n} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_n. \tag{2.3}$$

This series is absolutely convergent if $k > n+1$ and it becomes a Siegel modular form of weight k for Γ_n : $E_k^{(n)} \in [\Gamma_n, k]$. Here we call this *the Siegel-Eisenstein series of degree n and weight k*. We write the Fourier expansion of $E_k^{(n)}$ by

$$E_k^{(n)}(Z) = \sum_{0 \le T \in \Lambda_n} a_k^{(n)}(T) \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)]. \tag{2.4}$$

It is known that any Fourier coefficient $a_k^{(n)}(T)$ is rational ([Si]). The explicit formula of $a_k^{(n)}(T)$ was studied by several authors ([Kau], [M], [Kat]). For later purpose, we shall introduce an abbreviation. For $T = (t_{ij}) \in \Lambda_n$ and $Z = (z_{ij}) \in \mathbb{H}_n$, we write

$$q^T := \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)] = \prod_{i<j} q_{ij}^{2t_{ij}} \prod_{i=1}^{n} q_i^{t_i}, \tag{2.5}$$

where $q_{ij} := \exp(2\pi\sqrt{-1}\, z_{ij})$, and $q_i = q_{ii}$, $t_i = t_{ii}$. So the Fourier expansion (2.1) can be rewritten as

$$f = \sum_{0 \le T \in \Lambda_n} a_f(T)\, q^T \in \mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \cdots, q_n]],$$

namely, f is regarded as an element of the formal power series ring $\mathbb{C}[q_{ij}, q_{ij}^{-1}]$ $[[q_1, \cdots, q_n]]$.

3 Bernoulli numbers and generalized Bernoulli numbers.

In this section we review some of the basic facts about Bernoulli numbers and generalized Bernoulli numbers. The ordinary *Bernoulli numbers* B_m are defined by

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{3.1}$$

As is well known, certain special values of the Riemann zeta function can be represented by the Bernoulli numbers: for any even positive integer m, we have

$$\zeta(1-m)=-\frac{B_m}{m}. \tag{3.2}$$

THEOREM 3.1 *(1) (Kummer) If m and n are positive even integers with $m \equiv n \pmod{p^{e-1}(p-1)}$ and $n \not\equiv 0 \pmod{p-1}$, then*

$$\left(1-p^{m-1}\right)\frac{B_m}{m} \equiv \left(1-p^{n-1}\right)\frac{B_n}{n} \pmod{p^e}. \tag{3.3}$$

(cf. [W], §5.3, Corollary 5.14).
(2) (von Staudt-Clausen) Let m be even and positive. Then

$$B_m+\sum_{p-1 \mid m} \frac{1}{p} \in \mathbb{Z}. \tag{3.4}$$

Consequently, pB_m is p-integral for all m and all p. (cf. [W], Theorem 5.10).
(3) (Carlitz) If $p^{e-1}(p-1) \mid m$, then we have

$$pB_m \equiv p-1 \pmod{p^e}. \tag{3.5}$$

(cf. [W], p.86, 5.11 (b)).
(4) Let $p>3$ be a prime number such that $p \equiv 3 \pmod 4$. Then we have

$$B_{\frac{p+1}{2}} \equiv -\frac{h(-p)}{2} \not\equiv 0 \pmod{p}. \tag{3.6}$$

(cf. [BS], Chap.5, §8, Problem 4 and [W], p.86, Exercise 5.9).

Let χ be a Dirichlet character of conductor $f=f_\chi$. *The generalized Bernoulli numbers* $B_{m,\chi}$ are defined by

$$\sum_{a=1}^{f} \frac{\chi(a)\, t\, e^{at}}{e^{ft}-1}=\sum_{m=0}^{\infty} B_{m,\chi} \frac{t^m}{m!}. \tag{3.7}$$

Note that $B_{m,\chi^0}=B_m$ (χ^0 : the principal character) except for $m=1$, where we have $B_{1,\chi^0}=\frac{1}{2}$, $B_1=-\frac{1}{2}$.
Let $L(s;\chi)$ be the Dirichlet L-function belonging to a Dirichlet character χ:

$$L(s;\chi):=\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}. \tag{3.8}$$

Then, for any integer $m \geq 1$, we have

$$L(1-m;\chi) = -\frac{B_{m,\chi}}{m} \tag{3.9}$$

(e.g. cf. [I], §2, Theorem 1). In the following, we shall state Carlitz's result about generalized Bernoulli numbers in the case that χ is quadratic.

THEOREM 3.2 (Carlitz [Ca]) *Suppose that χ is a quadratic Dirichlet character of conductor f_χ.*
(1) If $\chi \neq \chi^0$, then $f_\chi B_{m,\chi}$ is a rational integer for every $m \geq 0$ and if f_χ is not a power of a prime, then even $\frac{1}{m}B_{m,\chi}$ is a rational integer.
(2) If p is a rational prime such that $p^e \mid m$ but $p \nmid f_\chi$, then p^e divides the numerator of $B_{m,\chi}$. If f_χ is divisible by at least two primes and p is arbitrary prime, then again p^e divides the numerator of $B_{m,\chi}$.
(3) Suppose that $f_\chi = p$ is an odd prime, and $p^{e-1} \parallel m$. Then

$$pB_{m,\chi} \equiv p-1 \pmod{p^e} \tag{3.10}$$

if $j(p-1) = 2m$ for some odd j.

REMARK. The original form of above statement (3) is as follows ([Ca], Theorem 3). Assume that $f_\chi = p$ is an odd prime and $p^{e-1} \parallel m$. Let $\wp$ be a prime ideal in $\mathbb{Q}(\chi)$ defined by

$$\wp = (p, 1-\chi(g)\, g^m)\,,$$

where g is a primitive root mod p. If $\wp \neq (1)$, then

$$pB_{m,\chi} \equiv p-1 \pmod{\wp^e}\,.$$

In our case, χ is quadratic, namely, $\mathbb{Q}(\chi) = \mathbb{Q}$. Obviously, if $j(p-1) = 2m$ for some odd j, then

$$\chi(g)\, g^m \equiv 1 \pmod{p}\,.$$

Therefore, Theorem 3.2, (3) is a special case of Carlitz's result.

4 Fourier coefficients of Siegel-Eisenstein series.

In this section, we shall introduce some explicit formulas of Fourier coefficient $a_k^{(n)}(T)$ of Siegel-Eisenstein series in the case $n \leq 2$.
It is well known that $a_k^{(1)}(t)$ $(4 \leq k \in 2\mathbb{Z})$ is given as follows:

$$a_k^{(1)}(t) = \begin{cases} -\dfrac{2k}{B_k}\sigma_{k-1}(t) & \text{if } \ t > 0\,, \\ 1 & \text{if } \ t = 0\,, \end{cases} \tag{4.1}$$

where $\sigma_m(t) := \sum_{0<d|t} d^m$.
In the case $n = 2$, G. Kaufhold [Kau] and H. Maass [M] gave explicit formulas. Here we introduce a description of $a_k^{(2)}$ by M. Eichler and D. Zagier [EZ] in

which they used Cohen's function $H(r, N)$.
Let r and N be non negative integers with $r \geq 1$. For $N \geq 1$, we define

$$h(r,N) := \begin{cases} (-1)^{[\frac{r}{2}]}(r-1)!\, N^{r-\frac{1}{2}} 2^{1-r} \pi^{-r} L(r; \chi_{(-1)^r N}) & \\ & \text{if} \quad (-1)^r N \equiv 0 \,\text{or}\, 1 \pmod 4, \\ 0 & \text{if} \quad (-1)^r N \equiv 2 \,\text{or}\, 3 \pmod 4, \end{cases}$$

where $L(s; \chi)$ is the Dirichlet L-function and we write χ_D for the character $\chi_D(d) = \left(\frac{D}{d}\right)$. Moreover, for $N \in \mathbb{R}$, we define

$$H(r,N) := \begin{cases} \sum_{d^2 | N} h\left(r, \frac{N}{d^2}\right) & \text{if} \quad (-1)^r N \equiv 0 \,\text{or}\, 1 \pmod 4,\ N > 0, \\ \zeta(1-2r) & \text{if} \quad N = 0, \\ 0 & \text{otherwise}. \end{cases}$$

The above defined function $H(r, N)$ is called *Cohen's function.* It is known that $H(r, N)$ has the following description.

Lemma 4.1 **([Co], p.273, c))** *If we set $(-1)^r N = Df^2$ with D discriminant of a quadratic field, then we have*

$$H(r,N) = L(1-r;\chi_D) \sum_{0<d|f} \mu(d)\, \chi_D(d)\, d^{r-1} \sigma_{2r-1}\left(\frac{f}{d}\right), \tag{4.2}$$

where $\mu(d)$ is the Möbius function.

Returning to the formula $a_k^{(2)}(T)$, for $O_2 \neq T \in \Lambda_2$ (cf. (2.2)), we define

$$\varepsilon(T) := \max\{l \in \mathbb{N} \mid l^{-1} T \in \Lambda_2\}. \tag{4.3}$$

Theorem 4.2 **([EZ], p.80, Corollary 2)** *If $0 \leq T \in \Lambda_2$ $(T \neq O_2)$, then*

$$a_k^{(2)}(T) = \frac{4k(k-1)}{B_k \cdot B_{2k-2}} \sum_{0<d|\varepsilon(T)} d^{k-1} H\left(k-1, \frac{\det(2T)}{d^2}\right). \tag{4.4}$$

Especially, if $\operatorname{rank} T = 1$, *then*

$$a_k^{(2)}(T) = -\frac{2k}{B_k} \sum_{0<d|\varepsilon(T)} d^{k-1} = -\frac{2k}{B_k} \sigma_{k-1}(\varepsilon(T)). \tag{4.5}$$

Remark. It should be noted that the factor $4k(k-1)/B_k \cdot B_{2k-2}$ in (4.4) is missing in the original formula of Eichler and Zagier.

By using (4.2), we can rewrite the formula (4.4). For $0 < T \in \Lambda_2$, we write

$$-\det(2T) = D(T) \cdot f(T)^2, \tag{4.6}$$

where $D(T)$ is the discriminant of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-\det(2T)}\right)$ and $f(T) \in \mathbb{N}$. It is quite obvious that the number $f(T)$ is divisible by $\varepsilon(T)$: $\varepsilon(T) \mid f(T)$.

Corollary 4.3 **(Explicit formula of $a_k^{(2)}(T)$)** *For $0 < T \in \Lambda_2$, we have*

$$
(4.7)\qquad
\begin{aligned}
a_k^{(2)}(T) &= -\frac{4k \cdot B_{k-1,\chi_{D(T)}}}{B_k \cdot B_{2k-2}} F_k(T), \\
F_k(T) &= \sum_{0<d|\varepsilon(T)} d^{k-1} \sum_{0<f|\frac{f(T)}{d}} \mu(f)\,\chi_{D(T)}(f)\, f^{k-2}\, \sigma_{2k-3}\left(\frac{f(T)}{fd}\right).
\end{aligned}
$$

5 p-adic Eisenstein series.

As we mentioned in Introduction, J. P. Serre developed the theory of p-adic modular form and applied it to the construction of p-adic zeta function. The p-adic Eisenstein series is a typical example of p-adic modular form. In this section, we shall briefly review Serre's theory.
In the following, for simplicity, we assume that p is an odd prime. Put

$$X_m := \mathbb{Z}/p^{m-1}(p-1)\mathbb{Z} = \mathbb{Z}/p^{m-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}, \quad m \geq 1.$$

Then $\{X_m\}$ forms a projective system. Let X be the limit of this system:

$$(5.1)\qquad X := \varprojlim X_m = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z},$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.
The p-adic modular form

$$(5.2)\qquad f = \sum_{t=0}^{\infty} a(t)\, q^t \in \mathbb{Q}_p[[q]]$$

is defined as the limit of a sequence of modular forms $\{f_m\}$ with rational Fourier coefficients. The limit means the following. Let v_p be the valuation on $\mathbb{Q}_p$ (the field of p-adic numbers) normalized as $v_p(p) = 1$. We denote by

$$f_m = \sum_{t=0}^{\infty} a^{(m)}(t)\, q^t \in \mathbb{Q}[[q]]$$

the Fourier expansion of f_m. The convergence $\lim_{m\to\infty} f_m = f$ means that

$$v_p(f - f_m) := \inf_t v_p(a(t) - a^{(m)}(t)) \to +\infty \quad (m \to \infty).$$

We denote by $\{k_m\} \subset 2\mathbb{Z}$ the weight of $\{f_m\}$. Serre [Se] showed that $\{k_m\}$ has the limit $\boldsymbol{k}$ in X. This element $\boldsymbol{k} \in X$ is called *the weight* of p-adic modular form f. The p-adic Eisenstein series (in the sense of Serre) is defined as follows. Put

$$G_k := \frac{1}{2}\zeta(1-k)E_k^{(1)} = -\frac{B_k}{2k}\, E_k^{(1)},$$

where $E_k^{(1)}$ is the Siegel-Eisenstein series of degree 1 and weight k $(4 \leq k \in 2\mathbb{Z})$. By (4.1), G_k has a Fourier expansion of the form

$$G_k = -\frac{B_k}{2k} + \sum_{t=1}^{\infty} \sigma_{k-1}(t)\, q^t \in \mathbb{Q}[[q]]\,.$$

Assume that $k \in X$. For an integer $t \geq 1$, we can define a p-adic integer $\sigma_{k-1}^*(t)$ by

$$\sigma_{k-1}^*(t) := \sum_{\substack{0<d|t \\ (d,p)=1}} d^{k-1}.$$

If $k \in X$ is even, then we can choose a sequence of integers $\{k_m\}$ $(4 \leq k_m \in 2\mathbb{Z})$ such that $k_m \to k \in X$ and $|k_m| \to +\infty$ where $|\cdot|$ is the ordinary absolute value. For this $\{k_m\}$, we have

$$\lim_{m\to\infty} \sigma_{k_m-1}(t) = \sigma_{k-1}^*(t)$$

in $\mathbb{Z}_p$. *The p-adic Eisenstein series (of degree 1) and weight $k \in X - \{0\}$ is* defined by

$$G_k^* = \lim_{m\to\infty} G_{k_m}. \tag{5.3}$$

Namely,

$$G_k^* = \frac{1}{2}\zeta^*(1-k) + \sum_{t=1}^{\infty} \sigma_{k-1}^*(t)\, q^t \in \mathbb{Q}_p[[q]]\,, \tag{5.4}$$

where the convergence of the constant term is guaranteed in [Se], 1.5, Cor. 2, and ζ^* is essentially the p-adic zeta function of Kubota and Leopoldt. Strictly speaking, if $(s,u) \in X = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ $((s,u) \neq 1)$, then

$$\zeta^*(s,u) = L_p(s;\omega^{1-u})\,, \tag{5.5}$$

where $L_p(s;\chi)$ is the p-adic L-function with character χ and ω is the Teichmüller character (e.g. cf. [I], p.18).

Example (Serre). Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$. If $k = \left(1, \frac{p+1}{2}\right) \in X$, then

$$G_k^* = \frac{1}{2}h(-p) + \sum_{t=1}^{\infty} \sum_{0<d|t} \left(\frac{d}{p}\right) q^t. \tag{5.6}$$

As mentioned before, $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$.

6 Main result.

One of the main purpose of this note is to give a generalization of the above-mentioned formula (5.6). It is interesting to us that the resulting formula has a simple form unexpectedly.

As was mentioned earlier, the Fourier expansion of Siegel modular form f can be written as

$$f = \sum_{0 \le T \in \Lambda_n} a_f(T)\, q^T \in \mathbb{C}[q_{ij}, q_{ij}^{-1}][[q_1, \cdots, q_n]] .$$

As an analogy of the degree one case, one can define the notion of *p-adic Siegel modular form* f as the limit of a sequence of ordinary Siegel modular forms $\{f_m\}$ with rational Fourier coefficients:

$$f = \sum_{0 \le T \in \Lambda_n} a(T)\, q^T \in \mathbb{Q}_p[q_{ij}, q_{ij}^{-1}][[q_1, \cdots, q_n]] ,$$

$$f_m = \sum_{0 \le T \in \Lambda_n} a^{(m)}(T)\, q^T \in \mathbb{Q}[q_{ij}, q_{ij}^{-1}][[q_1, \cdots, q_n]] ,$$

$$v_p(f - f_m) := \inf_{0 \le T \in \Lambda_n} v_p\left(a(T) - a^{(m)}(T)\right) \to +\infty \quad (m \to \infty) .$$

Our result is as follows:

Theorem 6.1 *Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$. If we put*

$$k_m := 1 + \frac{p-1}{2} \cdot p^{m-1} \in \mathbb{N} ,$$

then the sequence $\{k_m\}_{m=1}^{\infty}$ has the limit $k = \left(1, \frac{p+1}{2}\right) \in X$ and

$$\begin{aligned} E_k^* &:= \lim_{m \to \infty} \left(\frac{1}{2} \zeta(1 - k_m)\, E_{k_m}^{(2)} \right) \\ &= \frac{1}{2} h(-p) + \sum_{\substack{0 \le T \in \Lambda_n \\ D(T) = -p \text{ or } 0}} \operatorname{rank}(T) \sum_{0 < d \mid \varepsilon(T)} \left(\frac{d}{p} \right) q^T , \end{aligned} \tag{6.1}$$

where we understand $D(T) = 0$ if $\det(T) = 0$.

To prove this theorem, we prepare some lemma.

Lemma 6.2 *For non negative integers k, N, we define $S_k(N) := \sum_{a=1}^{N} a^k$. Then, for any prime $p > 3$ and integer $h \ge 1$, the following congruence relation holds:*

$$\frac{S_{k_m}(p^h)}{p^h} \equiv B_{k_m} \pmod{p^h} , \tag{6.2}$$

where B_{k_m} is the k_m-th Bernoulli number and k_m is the integer defined in Theorem 6.1.

PROOF. Let $B_n(x)$ be the n-th Bernoulli polynomial. The following identity is well known:

$$S_k(N) = \frac{1}{k+1}\left(B_{k+1}(N) - B_{k+1}(0)\right)$$

(e.g. cf. [I], p.15). Since

$$B_{k+1}(x) - B_{k+1}(0) = (k+1)\cdot B_k \cdot x + \binom{k+1}{2}\cdot B_{k-1}\cdot x^2 + \cdots,$$

we have

$$\frac{S_{k_m}(p^h)}{p^h} = B_{k_m} + \frac{k_m}{2}\cdot B_{k_m-1}\cdot p^h + \frac{k_m(k_m-1)}{2\cdot 3}\cdot B_{k_m-2}\cdot p^{2h} + \cdots.$$

The prime p does not appear in the denominator of B_{k_m-1} and appears at most once those of B_{k_m-j} $(j \geq 2)$. This shows (6.2). □

PROOF of Theorem 6.1. Put

$$E_{k_m} := \frac{1}{2}\zeta(1-k_m)\,E^{(2)}_{k_m}. \tag{6.3}$$

We write the Fourier expansion of E_{k_m} by

$$E_{k_m} = \sum_{0\leq T\in\Lambda_2} a^{(m)}(T)\,q^T \in \mathbb{Q}[q_{12}, q_{12}^{-1}][[q_1, q_2]]. \tag{6.4}$$

Moreover, put

$$a(T) := \begin{cases} \frac{1}{2}h(-p) & \text{if } T = O_2, \\ \sum_{0<d\mid\varepsilon(T)} \left(\frac{d}{p}\right) & \text{if } \operatorname{rank}(T) = 1, \\ 2\sum_{0<d\mid\varepsilon(T)} \left(\frac{d}{p}\right) & \text{if } \operatorname{rank}(T) = 2 \text{ and } D(T) = -p, \\ 0 & \text{otherwise}. \end{cases} \tag{6.5}$$

Our aim is to show the following:

$$\inf_{0\leq T\in\Lambda_2} v_p\left(a^{(m)}(T) - a(T)\right) \to +\infty \quad (m\to\infty). \tag{6.6}$$

As a first step, we shall show that

$$\lim_{m\to\infty} a^{(m)}(O_2) = \lim_{m\to\infty}\left(-\frac{B_{k_m}}{2k_m}\right) = \frac{1}{2}h(-p). \tag{6.7}$$

Although this is a part of the result (5.6), we shall give a direct proof. By Kummer's congruence (3.3),

$$\left(1-p^{k_m-1}\right)\frac{B_{k_m}}{k_m} \equiv \left(1-p^{k_l-1}\right)\frac{B_{k_l}}{k_l} \pmod{p^l}$$

for $m > l$ (note that $p > 3$). This means that the sequence $\{(1-p^{k_m-1})B_{k_m}/k_m\}$, hence $\{B_{k_m}/k_m\}$ converges in $\mathbb{Q}_p$. By Euler's criterion,

$$a^{k_m} \doteq \left(a^{\frac{p-1}{2}}\right)^{p^{m-1}} \cdot a \equiv \left(\frac{a}{p}\right) a \pmod{p^m}.$$

Hence we have

$$S_{k_m}(p^h) = \sum_{a=1}^{p^h} a^{k_m} \equiv \sum_{a=1}^{p^h} \left(\frac{a}{p}\right) a = \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) p^{h-1} \pmod{p^m} \tag{6.8}$$

for any positive integers m, h with $m > h$, equivalently,

$$\frac{S_{k_m}(p^h)}{p^h} \equiv \frac{1}{p}\left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \pmod{p^{m-h}}. \tag{6.9}$$

From this, we have

$$\lim_{m\to\infty} \frac{S_{k_m}(p^h)}{p^h} = \frac{1}{p}\left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \tag{6.10}$$

for any fixed integer h. Using (6.2), we obtain

$$\lim_{m\to\infty} \frac{B_{k_m}}{k_m} = \lim_{m\to\infty} B_{k_m} \equiv \lim_{m\to\infty} \frac{S_{k_m}(p^h)}{p^h} = \frac{1}{p}\left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \pmod{p^h}.$$

This shows

$$\lim_{m\to\infty} \frac{B_{k_m}}{k_m} = \frac{1}{p}\left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right). \tag{6.11}$$

From the general formula for $h(D)$ (D: fundamental discriminant), we get the following identity:

$$h(-p) = -\frac{1}{p}\left(\sum_{a=1}^{p-1} \chi_{-p}(a)\, a\right) = -\frac{1}{p}\left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) a\right) \tag{6.12}$$

(e.g. cf. [Z], §9, Satz 3). Combining (6.11) and (6.12), we get (6.7). The second step is to prove the following: for $T \neq O_2$,

$$a^{(m)}(T) \equiv a(T) \pmod{p^m}. \tag{6.13}$$

or equivalently,

$$\inf_{O_2 \neq T \in \Lambda_2} v_p\left(a^{(m)}(T) - a(T)\right) \geq m. \tag{6.14}$$

First assume that T is rank 1. In this case, by (4.5), we have

$$a^{(m)}(T)=-\frac{B_{k_m}}{2k_m}\cdot a^{(2)}_{k_m}(T)=\sigma_{k_m-1}(\varepsilon(T)).$$

Again by Euler's criterion, we obtain

(6.15)
$$a^{(m)}(T)=\sum_{0<d|\varepsilon(T)} d^{k_m-1}=\sum_{0<d|\varepsilon(T)} d^{\frac{p-1}{2}\cdot p^{m-1}}\equiv\sum_{0<d|\varepsilon(T)}\left(\frac{d}{p}\right)\pmod{p^m}.$$

Finally we assume that $T\in\Lambda_2$ is rank 2. By Corollary 4.3, $a^{(m)}(T)$ can be written as

(6.16)
$$a^{(m)}(T)=-\frac{B_{k_m}}{2k_m}\cdot a^{(2)}_{k_m}(T)=\frac{2B_{k_m-1,\chi_{D(T)}}}{B_{2k_m-2}}\cdot F_{k_m}(T),$$
$$F_{k_m}(T)=\sum_{0<d|\varepsilon(T)} d^{k_m-1}\sum_{0<f|\frac{f(T)}{d}}\mu(f)\,\chi_{D(T)}(f)\,f^{k_m-2}\sigma_{2k_m-3}\left(\frac{f(T)}{fd}\right).$$

We shall prove the following:

$$\text{(6.17)}\qquad \frac{B_{k_m-1,\chi_{D(T)}}}{B_{2k_m-2}}\equiv\left\{\begin{array}{ll}1 & \text{if}\quad D(T)=-p\\ 0 & \text{otherwise}\end{array}\right\}\pmod{p^m}.$$

By definition, the factor of Bernoulli numbers becomes

$$\frac{B_{k_m-1,\chi_{D(T)}}}{B_{2k_m-2}}=\frac{B_{\frac{p-1}{2}\cdot p^{m-1},\chi_{D(T)}}}{B_{(p-1)p^{m-1}}}.$$

Suppose that $D(T)\neq -p$. By Theorem 3.2, (1), (2) and (3.5), we have

$$B_{\frac{p-1}{2}\cdot p^{m-1},\chi_{D(T)}}\equiv 0\pmod{p^m},\quad pB_{(p-1)p^{m-1}}\equiv p-1\pmod{p^m}.$$

From these formulas, we get

$$\frac{B_{\frac{p-1}{2}\cdot p^{m-1},\chi_{D(T)}}}{B_{(p-1)p^{m-1}}}\equiv 0\pmod{p^m}.$$

Suppose that $D(T)=-p$. By (3.5) and Theorem 3.2, (3), we have

$$pB_{\frac{p-1}{2}\cdot p^{m-1},\chi_{-p}}\equiv p-1\pmod{p^m},\quad pB_{(p-1)p^{m-1}}\equiv p-1\pmod{p^m}.$$

From these formulas, we obtain

$$\frac{B_{\frac{p-1}{2}\cdot p^{m-1},\chi_{-p}}}{B_{(p-1)p^{m-1}}}\equiv 1\pmod{p^m},$$

and this completes the proof of (6.17). Next we shall show that, if $D(T) = -p$, then

(6.18) $$F_{k_m}(T) \equiv \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) \quad (\mathrm{mod}\ p^m).$$

In our case, we have $\chi_{D(T)}(a) = \chi_{-p}(a) = \left(\frac{a}{p}\right)$. Therefore

$$F_{k_m}(T) \equiv \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) \sum_{\substack{0<f|\frac{f(T)}{d} \\ (f,p)=1}} \mu(f)\, f^{-1} \sigma^*_{-1}\left(\frac{f(T)}{fd}\right) \quad (\mathrm{mod}\ p^m),$$

where $\sigma^*_{-1}(l) = \sum_{0<d|l,(d,p)=1} d^{-1}$ (cf. §5). To prove (6.18), it suffices to show that

(6.19) $$\sum_{\substack{0<f|\frac{f(T)}{d} \\ (f,p)=1}} \mu(f)\, f^{-1} \sigma^*_{-1}\left(\frac{f(T)}{fd}\right) = 1$$

for any d with $d \mid \varepsilon(T)$. In general, we can prove

(6.20) $$\sum_{\substack{0<l|m \\ (l,p)=1}} \mu(l)\, l^{-1} \sigma^*_{-1}\left(\frac{m}{l}\right) = 1$$

for any $m \in \mathbb{N}$. For any $m \in \mathbb{N}$ with $p^e \parallel m$, we put $m_0 := m/p^e = p_1^{e_1} \cdots p_r^{e_r}$ (p_i : prime $\neq p$). Then

$$\begin{aligned} \sum_{\substack{0<l|m \\ (l,p)=1}} \mu(l)\, l^{-1} \sigma^*_{-1}\left(\frac{m}{l}\right) &= \sum_{0<l|m} \mu(l)\, l^{-1} \sigma_{-1}\left(\frac{m_0}{l}\right) \\ &= \prod_{i=1}^{r} \left(\sum_{0<l|p_i} \mu(l)\, l^{-1} \sigma_{-1}\left(\frac{p_i}{l}\right) \right). \end{aligned}$$

The inner sum of the last formula is trivially equal to 1. This shows (6.20). Combining (6.17) and (6.18), we obtain

$$a^{(m)}(T) \equiv \left\{ \begin{array}{ll} 2 \displaystyle\sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) & \text{if } D(T) = -p \\ 0 & \text{otherwise} \end{array} \right\} \quad (\mathrm{mod}\ p^m).$$

This proves (6.13). If we put $b_m := v_p\left(a^{(m)}(O_2) - a(O_2)\right)$, then, by (6.5) and (6.7), we have $b_m \to +\infty$ $(m \to \infty)$. Therefore we obtain

$$\inf_{0 \leq T \in \Lambda_2} v_p\left(a^{(m)}(T) - a(T)\right) \geq \min(m, b_m) \to +\infty \quad (m \to \infty).$$

This showa (6.6) and completes the proof of Theorem 6.1. □

7 Reduction mod p of Fourier coefficient of Siegel-Eisenstein series.

By similar argument used in §6, we can present an additional formula for the Fourier coefficient of Siegel-Eisenstein series of degree 2.
The following result is due to Yamaguchi.

THEOREM 7.1 (Yamaguchi [Y]) *Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$. For any $0 < T \in \Lambda_2$ with $f(T) = 1$, we have*

$$a^{(2)}_{\frac{p+1}{2}}(T) \equiv -\frac{4p\, B_{\frac{p-1}{2},\chi_{D(T)}}}{h(-p)} \pmod p \tag{7.1}$$

(for the definition of $f(T)$, see (4.6)).

REMARK. The right-hand side does not necessarily vanish because there is a possibility that prime p appears in the denominator of $B_{\frac{p-1}{2},\chi_{D(T)}}$.
We can genralize the above result.

THEOREM 7.2 *Let $p > 3$ be a prime number such that $p \equiv 3 \pmod 4$. For any $0 < T \in \Lambda_2$, we have*

$$a^{(2)}_{\frac{p+1}{2}}(T) \equiv \frac{4\alpha_T}{h(-p)} \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) \pmod p, \tag{7.2}$$

where

$$\alpha_T := \begin{cases} 1 & \text{if } D(T) = -p, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. By Corollary 4.3, we can write as

$$a^{(2)}_{\frac{p+1}{2}}(T) = -\frac{2(p+1)\cdot B_{\frac{p-1}{2},\chi_{D(T)}}}{B_{\frac{p+1}{2}}\cdot B_{p-1}} \cdot F_{\frac{p+1}{2}}(T).$$

Recall

$$B_{\frac{p+1}{2}} \equiv -\frac{h(-p)}{2} \not\equiv 0 \pmod p, \quad \text{(Theorem 3.1, (4))}.$$

This implies

$$a^{(2)}_{\frac{p+1}{2}}(T) \equiv \frac{4(p+1)B_{\frac{p-1}{2},\chi_{D(T)}}}{h(-p)\cdot B_{p-1}} \cdot F_{\frac{p+1}{2}}(T) \pmod p. \tag{7.3}$$

First suppose that $D(T) \neq -p$. In this case, p does not appear in the denominator of $B_{\frac{p-1}{2},\chi_{D(T)}}$ (cf. Theorem 3.2, (1)). Then, by the theorem of von Staudt-Clausen (Theorem 3.1, (2)), the right-hand side of (7.3) is divisible by p. Secondly suppose that $D(T) = -p$. In this case, we have

$$pB_{p-1} \equiv -1 \pmod p, \quad pB_{\frac{p-1}{2},\chi_{-p}} \equiv -1 \pmod p$$

(cf. (3.5), (3.10)). Therefore, we get

$$\frac{B_{\frac{p-1}{2},\chi_{-p}}}{B_{p-1}} \equiv 1 \pmod{p}. \tag{7.4}$$

So we can rewrite (7.3) as

$$a^{(2)}_{\frac{p+1}{2}}(T) \equiv \frac{4\alpha_T}{h(-p)} F_{\frac{p+1}{2}}(T) \pmod{p}.$$

We shall show

$$F_{\frac{p+1}{2}}(T) \equiv \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) \pmod{p}. \tag{7.5}$$

The proof of this formula is the same as that of (6.18). In fact, we have

$$\begin{aligned} F_{\frac{p+1}{2}}(T) &= \sum_{0<d|\varepsilon(T)} d^{\frac{p-1}{2}} \sum_{0<f|\frac{f(T)}{d}} \mu(f)\,\chi_{-p}(f)\, f^{\frac{p-3}{2}} \sigma_{p-2}\left(\frac{f(T)}{fd}\right) \\ &\equiv \sum_{0<d|\varepsilon(T)} \left(\frac{d}{p}\right) \sum_{\substack{0<f|\frac{f(T)}{d} \\ (f,p)=1}} \mu(f)\, f^{-1} \sigma^*_{-1}\left(\frac{f(T)}{fd}\right) \pmod{p}. \end{aligned}$$

We can show by (6.20) that the inner sum is equal to 1. This proves (7.5), and consequently, we get (7.2). □

References

[BS] Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, New York, 1967.

[Ca] L. Carlitz, *Arithmetic properties of generalized Bernoulli numbers*, J. Reine Angew. Math. 202 (1959), 174–182.

[Co] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. 217 (1975), 271–285.

[EZ] M. Eichler and D. B. Zagier, *The Theory of Jacobi Forms*, Birkhäuser, Boston Basel Stuttgart, 1985.

[I] K. Iwasawa, *Lectures on p-adic L-functions*, Annals of Math. Studies 74, Princeton Univ. Press, Princeton N. J., 1972.

[Kat] H. Katsurada, *An explicit formula for Siegel series*, preprint 1997.

[Kau] G. Kaufhold, *Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades*, Math. Ann. 137 (1959), 454–476.

[M] H. Maass, *Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades*, Mat. Fys. Medd. Dan. Vid. Selsk 34 (1964), 1–25.

[Se] J. -P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Modular functions of one variable III, 191–268, Lecture Notes in Math. 350, Springer Verlag, 1973.

[Si] C. L. Siegel, *Einführung in die Theorie der Modulfunktionen n-ten Grades*, Math. Ann. 116 (1939), 617–657.

[W] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer Verlag, New York Heidelberg Berlin, 1982.

[Y] I. Yamaguchi, *On Bernoulli numbers and its application*, 1996.

[Z] D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer Verlag, Berlin Heidelberg New York, 1981.

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